LECTURE 10

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1. Projective varieties

For $0 \le i \le n$, let $U_i = \mathbb{P}_k^n - V(X_i) = \{[X_0 : \cdots : X_n] | X_i \neq 0\}$. Sometimes U_i is called a fundamental open set of $\mathbb{P}_k^n$. We have a bijection

$$\phi_i : \mathbb{A}_k^n \xrightarrow{\sim} U_i$$

$$(y_1, \cdots, y_n) \mapsto [y_1 : y_2 : \cdots : y_{i-1} : 1 : y_i : \cdots : y_n].$$

The inverse ϕ_i^{-1} is given by $[x_0 : \cdots : x_n] \mapsto (x_0/x_i, \cdots, x_{i-1}/x_i, x_{i+1}/x_i, \cdots, x_n/x_i)$. There is a precise sense in which the above map is an isomorphism. We will not dwell on that at this point. Intuitively these identifications should be thought of as charts as for manifolds. Note that $\mathbb{P}_k^n$ is covered by $U_0, \cdots, U_n$. So we may say that $\mathbb{P}_k^n$ is covered by $n+1$ copies of the n-dimensional affine space.

Let $V \subset \mathbb{P}_k^n$ be a projective algebraic set with homogeneous ideal $I(V)$. Then $\phi_i^{-1}(V \cap U_i) \subset \mathbb{A}_k^n$ is an affine algebraic set, with ideal

$$\{f(Y_1, \cdots, Y_{i-1}, 1, Y_i, \cdots, Y_n) | f \in I(V)\}.$$

We will abuse notation to denote this affine algebraic set also by $V \cap U_i$. Note that since the U_i's cover $\mathbb{P}_k^n$, the $V \cap U_i$'s cover V. We may say that V is covered by $n+1$ affine algebraic sets. Also note that if V is defined over k, then so is each $V \cap U_i$ as an affine algebraic set in $\mathbb{A}_k^n$.

Example 1.1. Consider $V = V(X^2 + Y^2 - 3Z^2) \subset \mathbb{P}_k^3$. $V \cap U_X = V(X^2 + Y^2 - 3) \subset \mathbb{A}_k^2 = \{(x, y)\}$, $V \cap U_Y = V(X^2 + 1 - 3Z^2) \subset \mathbb{A}_k^2 = \{(x, z)\}$.

Conversely, given an affine algebraic set $V \subset \mathbb{A}_k^n$ and an $i, 0 \le i \le n$, we can produce a projective algebraic set, denoted by $\overline{\phi_i(V)}$, called the closure of $\phi_i(V)$, defined as follows. Firstly, for any polynomial $f(Y_1, \cdots Y_n)$ in n variables, we can produce a homogeneous polynomial f^* in $n+1$ variables, defined by

$$f^*(X_0, \cdots, X_n) = X_i^{\deg f} f(X_0/X_i, \cdots, X_{i-1}/X_i, X_{i+1}/X_i, \cdots, X_n/X_i).$$

We say f^* is the homogenization of f by adding the i-th variable.

Example 1.2. Let $f(X, Y, Z) = X^2 + Y - Z^3$. We homogenize f by adding the variable W. We have

$$f^*(X, Y, Z, W) = W^3 f(X/W, Y/W, Z/W) = W^3(X^2/W^2 + Y/W - Z^3/W^3) = X^2W + YW^2 - Z^3.$$

Let I' be the homogeneous ideal of $\bar{k}[X_0, \cdots, X_n]$ generated by $\{f^* | f \in I(V)\}$. We define $\overline{\phi_i(V)} := V(I') \subset \mathbb{P}_k^n$.

Example 1.3. Consider $\phi_W : \mathbb{A}_k^3 = \{(x, y, z)\} \xrightarrow{\sim} U_W = \mathbb{P}_{k,x,y,z,w}^3 - V(W)$. Then $\overline{\phi_W(V(X^2 + Y - Z^3))} = V(X^2W + YW^2 - Z^3)$.

Proposition 1.4. *(1) Let $V \subset \mathbb{A}^n_k$ be an affine algebraic variety. Then $V' = \overline{\phi_i(V)} \subset \mathbb{P}^n_k$ is a projective variety. We have $V = V' \cap U_i$, and V is defined over k if and only if V' is defined over k.*
(2) Let $V' \subset \mathbb{P}^n_k$ be a projective variety. Then $V = V' \cap U_i$ is an affine variety. Either $V = \emptyset$ or $V' = \overline{\phi_i(V)}$.

In this way we can uniquely associate a projective variety V' to an affine variety V. We will often use this construction tacitly. We also call $V' - V$ the points at infinity.

Example 1.5. Consider the affine varieties $V_1 = V(X^2 - Y^2 - 1), V_2 = V(X - Y) \subset \mathbb{A}^2_k$. We have $V_1 \cap V_2 = \emptyset$. Consider $\phi_Z : \mathbb{A}^2_k \to U_Z \subset \mathbb{P}^2_k = \{[X : Y : Z]\}$. We have $V_1' = \overline{\phi_Z(V_1)} = V(X^2 - Y^2 - Z^2), V_2' = \overline{\phi_Z(V_2)} = V(X - Y) \subset \mathbb{P}^2_k$. We have $V_1' \cap V_2' = \{[1 : 1 : 0]\}$. Note this point is not in U_Z. This makes precise the intuitive idea that V_1 and V_2 intersect at a point at infinity. Moreover, we have $V_1' \cap U_X = V(1 - Y^2 - Z^2), V_2' \cap U_X = V(1 - Y)$. Thus in the $Y - Z$-affine space, the intersection behaves the same as the intersection of a circle with a line tangent to it.

Example 1.6. Suppose $\operatorname{char} k \neq 2$. Let $V = V(aX^2 + bXY + cY^2 + dX + eY + f) \subset \mathbb{A}^2_k$ be a non-degenerate conic curve (i.e. such that it is not a union of points and lines). Consider its projective closure $V' = V(aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2) \subset \mathbb{P}^3_k$. It can be shown that after a linear coordinate change $(X, Y, Z) = (X', Y', Z')A, A \in \mathrm{GL}_3(\bar{k})$, we can put V' into the form $V' = V(X^2 + Y^2 + Z^2)$. This is to say, all the conic curves are equivalent as projective curves. In fact, as we will see later, they are all isomorphic to $\mathbb{P}^1$. This is one of the first merits of projective geometry that fascinated early 19th century geometers so much.

Definition 1.7. Let $V \subset \mathbb{P}^n_k$ be a projective variety. Suppose $V \cap U_i \neq \emptyset$. Define

$$\dim V := \dim V \cap U_i$$

$$\bar{k}(V) := \bar{k}(V \cap U_i).$$

In case V is defined over k, define

$$k(V) = k(V \cap U_i).$$

For any $p \in V \cap U_i$, define the local ring $\mathcal{O}_{V,p}$ at p and smoothness of V at p according to the affine variety $V \cap U_i$.

Remark 1.8. The above definition does not depend on the choice of i, as long as $V \cap U_i \neq \emptyset$. Moreover, if we change the coordinates on $\mathbb{P}^n$ by an element in $\mathrm{GL}_{n+1}(\bar{k})$, the definition still stays invariant.

Example 1.9. Suppose $\operatorname{char} k \neq 2, 3$. Consider $V = V(Y^2Z - X^3 + XZ^2)$. It is the projective closure of the affine curve $Y^2 = X^3 - X$ that we have seen is nonsingular. Is V also nonsingular at the points at infinity? Set $Z = 0$ in the equation defining V, we see $X^3 = 0$. Hence V has only one point at infinity, $p = [0 : 1 : 0]$. To study the smoothness of V, we look at $V \cap U_Y$ which contains p. $V \cap U_Y = V(Z - X^3 + XZ^2) \subset \mathbb{A}^2_{k,x,z}$. The partial X and Z derivatives of $Z - X^3 + XZ^2$ are $-3X^2 + Z^2$ and $1 + 2XZ$, resp. $1 + 2XZ$ does not vanish at the point $X = Z = 0$. Hence V is nonsingular at $[0 : 1 : 0]$.

Example 1.10. Consider $V = V(X^2+Y^2+Z^2) \subset \mathbb{P}^1_k$. $V \cap U_Z = V(X^2+Y^2+1) \subset \mathbb{A}^2_{k,x,y}$, $V \cap U_Y = V(X^2+1+Z^2) \subset \mathbb{A}^2_{k,x,z}$. $k(V \cap U_Z) = k(X,Y)/(X^2+Y^2+1), k(V \cap U_y) = k(X,Z)/(X^2+1+Z^2)$. The natural isomorphism $k(V \cap U_Z) \xrightarrow{\sim} k(V \cap U_Y)$ is given by first homogenizing and then dehomogenizing. For example, given $f(X,Y) = X/Y^2 \in k(V \cap U_Z)$, we get $f(X/Z, Y/Z) = XZ/Y^2$. Set $g(X,Z) = XZ/Y^2|_{Y=1} = XZ \in k(V \cap U_y)$.

Let $V \subset \mathbb{P}^n_k$ be a projective variety defined over k. We can interpret the function field $k(V)$ as follows: The elements in $k(V)$ are rational functions in $n+1$ variables of the form $F = f(X_0, \cdots, X_n)/g(X_0, \cdots, X_n), g \neq 0$, where f and g are homogeneous polynomials of the same degree. Two elements $f/g, f_1/g_1$ are viewed as the same if $fg_1 - f_1 g \in I(V) \subset \bar{k}[X_0, \cdots, X_n]$. We see that F defines a k-valued function on V except where g vanishes. If $p \in V - V(g)$, we say F is regular at p. We see that F is regular at p if and only if $F \in \mathcal{O}_{V,p} \subset \bar{k}(V)$.

Exercise 1.11. Let $V \subset \mathbb{P}^n_k$ be a projective variety. Let $f: V \to \bar{k}$ be a function such that for all i, $f|_{V \cap U_i} : V \cap U_i \to \bar{k}$ is given by an element in $\bar{k}[V \cap U_i]$. Show that f has to be a constant function.

2. Maps between algebraic varieties

To get a complete theory of morphisms between algebraic varieties, we would have to talk more about the Zariski topology, which fortunately (or unfortunately?) is not required for our purpose. We are satisfied to only use the notion of rational maps between projective varieties.

Let $V \subset \mathbb{P}^n_k$ be a projective variety. Recall that a rational function $F \in \bar{K}(V)$ can be interpreted as a quotient f/g of two homogeneous polynomials of the same degree, and in particular it defines a $\bar{k}$ valued function on V except where g vanishes.

Definition 2.1. A rational map $F: V \to \mathbb{P}^m_k$ is an element of $\mathbb{P}^m_{\bar{k}(V)}$, i.e. it is an $m+1$ tuple $[F_0 : \cdots : F_m]$, with $F_i \in \bar{k}(V)$ not all zero, and we equate $[F_0 : \cdots : F_m] = [GF_0 : \cdots : GF_m]$ for any $G \in \bar{k}(V) - \{0\}$.

Let $F = [F_0 : \cdots : F_m] : V \to \mathbb{P}^m_k$ be a rational map. Suppose $p \in V$ is such that all the F_i's are regular at p, then we can evaluate F at p to get a point $F(p)$ in $\mathbb{P}^m_k$. Moreover, even if some or all of the F_i's are not regular at p, it may happen that there exists $G \in \bar{k}(V) - \{0\}$ such that for each i, GF_i is regular at p. Also assume not all $(GF_i)(p) = 0$. In this case, we may replace $[F_0 : \cdots : F_n]$ by $[GF_0 : \cdots : GF_m]$ and evaluate at p to get a point $F(p)$ in $\mathbb{P}^m_k$. We see that $F(p)$ is independent of the choice of G.

Definition 2.2. Let $F = [F_0 : \cdots : F_m] : V \to \mathbb{P}^m_k$ be a rational map. Let $p \in V$. We say F is regular at p, if there exists $G \in \bar{k}(V) - \{0\}$ such that for each i, GF_i is regular at p, and not all $(GF_i)(p) = 0$. In this case we define $F(p) \in \mathbb{P}^m_k$ by evaluating. If F is regular at all the points in V, we say F is a morphism.